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Complete analysis of bifurcations in the axial gyrostat problem

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Abstract. We analyse the phase flow evolution of the torque free asymmetric gyrostat motion. The gyrostat consists of a triaxial rigid body and a symmetric rotor spinning around one of the principal axis of inertia of the gyrostat. The problem is converted into a two parametric quadratic Hamiltonian with the phase space on the $\mathcal{S}^2$ sphere. As the parameters evolve, the appearance–disappearance of centres and saddle points is originated by a sequence of pitchfork bifurcations. When the gyrostat is axial symmetric, there are motions of the rotor that break the degeneracy through an oyster bifurcation while other motions simply shift the degeneracy along a minor circle.

1. Introduction

A gyrostat is a mechanical system $\mathcal{G}$ composed by a rigid body $\mathcal{P}$ (platform) and other bodies $\mathcal{R}$ (rotors) connected to it, in such a way that the motion of the rotors does not modify the distribution of masses of the gyrostat $\mathcal{G}$. This problem has been studied since the last century for modelling the rotation of the Earth [1]. For details, the reader is addressed to the book of Leimanis [2] and references therein.

More recently, the dynamics of the gyrostat, so-called dual-spin, has been an object of interest in astrodynamics, and it is used, for instance, for controlling the attitude dynamics of spacecrafts, and for stabilizing their rotations [3–10] and also [11] for further references.

The gyrostat in force free motion and with constant internal moments is an integrable case, and its solution is given in terms of elliptic functions (see e.g. [12]). However, this case is of great importance for it represents the unperturbed part of more complex models, such as the heavy gyrostat, or the gyrostat in a Newtonian force field. Even this model may represent nuclear physics problems [13] or optical problems, for instance, as it is proved by Holm and co-workers [14, 15], the equations of motion for the Stokes polarization parameters of a single optical beam in a nonlinear medium are analogous to the ones of the gyrostat. Thus, the better will be our knowledge of this case, the more will be our understanding of the perturbed problem.

It is well known that even in the torque free gyrostat motion, the phase flow depends on the magnitude of the principal moments of inertia, on the rotor moments and on its directions, and that some bifurcations occur. These bifurcations, or more precisely the unstable points (saddle points), are the seeds of chaos [16] and some attempts have been made in this direction of finding whether this model under some perturbations has chaotic dynamics [17].

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In a recent paper, Tong *et al* [18] considered an asymmetric gyrostat, with constant rotor moment along the biggest moment of inertia under the effect of the gravitational field, and by making use of the Melnikov theorem [19], they prove that the motion is chaotic in the sense of Smale's horseshoes when the angular momentum of the rotor is small. In order to apply the Melnikov theorem, one needs one homoclinic orbit of the unperturbed problem, that is, the gyrostat in torque free motion. Tong *et al* [18] make use of the Serret–Andoyer canonical variables to represent the phase flow, since they do not seem to be aware of the fact that, since the total angular moment is preserved, the topology of the phase space is of the sphere $\mathcal{S}^2$ (as it was already pointed out in [17, 20]) rather than the plane, and what is even worse, for the case they choose, bifurcations occur precisely at the poles of the $\mathcal{S}^2$ sphere, that are singular points in the Mercator representation $(\ell, L/G)$ they use.

The present paper aims to clarify what is the phase flow of an axial gyrostat in torque free motion, depending on the spins of the rotors, that are considered as parameters. When the gyrostat is asymmetric, the problem depends on two parameters, one for the principal moments of inertia, and the second one for the rotor moment. It is shown (sections 4–6) that when the spin of the rotor is along one axis of inertia, the phase flow bifurcates on the intersection of these axes with the $\mathcal{S}^2$ sphere through a sequence of two pitchfork bifurcations. In the case that the gyrostat is axially symmetric (section 7), the degeneracy due to this symmetry is either broken through an oyster bifurcation or shifted to a minor circle, depending on the spin axis of the rotor.

2. Integrals of the problem

Let us consider two orthonormal reference frames with origin O at the centre-of-mass of the gyrostat, one $\mathcal{S}$ fixed in the space $\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3$ and the other $\mathcal{B}$, the body frame $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$, fixed in the body. The attitude of $\mathcal{B}$ in $\mathcal{S}$ results in three rotations by means of the Euler angles (ϕ, ϑ, ψ).

The nutation angle ϑ $(0 \leqslant \vartheta \leqslant \pi)$ is defined by the dot product $\cos\vartheta = \boldsymbol{b}_3 \cdot \boldsymbol{s}_3$. The vector $\boldsymbol{l}$, the direction of the intersection of the space plane $(\boldsymbol{s}_1, \boldsymbol{s}_2)$ with the body plane $(\boldsymbol{b}_1, \boldsymbol{b}_2)$, is obtained by $\boldsymbol{l} = \boldsymbol{s}_3 \times \boldsymbol{b}_3/\sin\vartheta$. This vector is related with the axes $(\boldsymbol{s}_1, \boldsymbol{s}_2)$ by

$$\boldsymbol{l} = \cos\phi \boldsymbol{s}_1 + \sin\phi \boldsymbol{s}_2 \qquad 0 \leqslant \phi < 2\pi$$

where the angle ϕ, usually known as the precession angle, is the longitude of the node $\boldsymbol{l}$ reckoned from the space axis $\boldsymbol{s}_1$. By denoting ψ (with $0 \leqslant \psi < 2\pi$) the longitude of the body vector $\boldsymbol{b}_1$ reckoned from the node $\boldsymbol{l}$, this vector is expressed as the combination

$$\boldsymbol{l} = \cos\psi \boldsymbol{b}_1 - \sin\psi \boldsymbol{b}_2.$$

By means of the composite rotation (see [21] for details) $R = R(\phi, \boldsymbol{s}_3) \circ R(\vartheta, \boldsymbol{l}) \circ R(\psi, \boldsymbol{b}_3)$, the space frame $\mathcal{S}$ is mapped onto the body frame $\mathcal{B}$, and by means of the differential of R,

$$\mathrm{d}R = \mathrm{d}\phi\, \boldsymbol{s}_3 + \mathrm{d}\vartheta\, \boldsymbol{l} + \mathrm{d}\psi\, \boldsymbol{b}_3 \tag{1}$$

we are able to obtain the angular velocity. Indeed, let ω be the angular velocity vector of the frame $\mathcal{B}$ with respect to the $\mathcal{S}$. This vector is

$$\omega = \dot\phi \boldsymbol{s}_3 + \dot\vartheta \boldsymbol{l} + \dot\psi \boldsymbol{b}_3$$

and expressed in the body frame is

$$\omega = \omega_1 \boldsymbol{b}_1 + \omega_2 \boldsymbol{b}_2 + \omega_3 \boldsymbol{b}_3$$

and thus,

$$\omega_1 = \dot\phi \sin\vartheta \sin\psi + \dot\vartheta \cos\psi \qquad \omega_2 = \dot\phi \sin\vartheta \cos\psi - \dot\vartheta \sin\psi \qquad \omega_3 = \dot\phi \cos\vartheta + \dot\psi.$$

Let I_1, I_2, I_3 be the principal moments of inertia of the gyrostat, that without loss of generality we will assume $I_1 < I_2 < I_3$.

The angular momentum vector $\boldsymbol{G}$ of the body in its rotation about the origin O is computed straightforward through its definition

$$\boldsymbol{G} = \sum_i m_i \boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i$$

and it results, when expressed in the body frame

$$\begin{aligned}\boldsymbol{G} &= g_1\boldsymbol{b}_1 + g_2\boldsymbol{b}_2 + g_3\boldsymbol{b}_3\\ &= (I_1\omega_1 + h_1)\boldsymbol{b}_1 + (I_2\omega_2 + h_2)\boldsymbol{b}_2 + (I_3\omega_3 + h_3)\boldsymbol{b}_3\\ &= \boldsymbol{g}^{rb} + \boldsymbol{h}\end{aligned} \tag{2}$$

where the vector $\boldsymbol{h} = h_1\boldsymbol{b}_1 + h_2\boldsymbol{b}_2 + h_3\boldsymbol{b}_3$ is the rotor momentum, that is to say, is the moment of the relative motion of the rotors. Thus, the total angular moment is the sum of two parts, the moment of the entire system $\mathcal{G}$ considered as a rigid body ($\boldsymbol{g}^{rb}$), plus the moment of the relative motion of the rotors ($\boldsymbol{h}$).

In a similar way, and by direct computation, one gets for the kinetic energy

$$\tilde{T} = \tfrac{1}{2}\omega \cdot \boldsymbol{g}^{rb} + \omega \cdot \boldsymbol{h} + T_r \tag{3}$$

with T_r the kinetic energy of the rotor in its relative motion.

By applying the formula of the derivation with respect to a moving frame to the angular momentum, there results the Euler equations

$$\dot{\boldsymbol{G}} = \dot{\boldsymbol{g}}^{rb} + \dot{\boldsymbol{h}} + \omega \times \boldsymbol{g}^{rb} + \omega \times \boldsymbol{h} = \boldsymbol{M} \tag{4}$$

where $\boldsymbol{M}$ stands for the resulting moment of the external forces. In the absence of external forces, the right-hand side of this equation vanishes, which means that the absolute derivative of the angular momentum vector is zero, that is to say, this vector is constant in the space frame $\mathcal{S}$ and consequently, its norm. But the norm of a vector is invariant under the action of the $SO(3)$ group,

$$\|\boldsymbol{G}\|^2 = g_1^2 + g_2^2 + g_3^2 = (I_1\omega_1 + h_1)^2 + (I_2\omega_2 + h_2)^2 + (I_3\omega_3 + h_3)^2 = G^2 = \text{constant} \tag{5}$$

thus, the total angular momentum vector in the body frame describes a curve on the S^2 sphere of radius G

When the internal moment is constant (h_i = constant, $i = 1, 2, 3$), and if there are no external forces, equation (4) is reduced to

$$\dot{\boldsymbol{g}}^{rb} + \omega \times \boldsymbol{g}^{rb} + \omega \times \boldsymbol{h} = 0$$

and by making the dot product of this equation and ω, the following expression yields

$$\dot{\boldsymbol{g}}^{rb} \cdot \omega = 0$$

and by integration,

$$\tfrac{1}{2}\boldsymbol{g}^{rb} \cdot \omega = \text{constant}$$

which expanded in its components is,

$$\tfrac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) = \text{constant}. \tag{6}$$

One should note that in absence of external forces, the total kinetic energy (3) is not conserved, but the kinetic energy of the gyrostat considered as a rigid body (6) is.

In summary, the problem considered (let us recall that we are dealing with a gyrostat with constant rotor moment and under no external forces) has two first integrals, the total angular momentum

$$g_1^2 + g_2^2 + g_3^2 = G^2 = \text{constant} \tag{7}$$

and the kinetic energy (6) of the rigid body, that expressed in the components of the total angular momentum takes the form

$$\hat{T} = a_1 g_1^2 + a_2 g_2^2 + a_3 g_3^2 - 2a_1 g_1 h_1 + a_1 h_1^2 - 2a_2 g_2 h_2 + a_2 h_2^2 - 2a_3 g_3 h_3 + a_3 h_3^2$$

where the coefficients a_i are the inverse of the principal moments of inertia of the gyrostat; therefore, for a given craft, they are fixed. Without loss of generality we will assume along the paper that $a_1 > a_2 > a_3 > 0$. Only when the gyrostat be assumed to have axial symmetry $a_1 = a_2 > a_3 > 0$ or $a_1 > a_2 = a_3 > 0$. By transferring the constant terms $(\sum a_i h_i^2)$ of this expression to the left-hand side, we obtain

$$T = \tfrac{1}{2}(a_1 g_1^2 + a_2 g_2^2 + a_3 g_3^2) - (a_1 g_1 h_1 + a_2 g_2 h_2 + a_3 g_3 h_3). \tag{8}$$

Thus, the phase flow will be made of the contour levels of the quadric (8) on the sphere (7).

3. Hamiltonian of the problem

To build the Hamiltonian function, the classical way (see for instance [8, 18]) is to define the conjugate moments of the three coordinates (the Euler angles ϕ, ϑ, ψ) by taking the partial derivatives of the Lagrangian function, and with it, compute the Legendre transformation of the Lagrangian. Before computing the conjugate moments, we shall obtain the Hamiltonian. The kinetic energy (3) is made of the addition of a pure quadratic term $(\frac{1}{2}\omega \cdot \boldsymbol{g}^{rb})$ in the velocities (and hence in the derivatives of the Euler angles), a linear part $(\omega \cdot \boldsymbol{h})$, since $\boldsymbol{h}$ does not depend on the Euler angles, and a pure function of the time $(T_r(t))$. By virtue of the Euler theorem for homogeneous functions, and by denoting $\boldsymbol{q} = (\phi, \vartheta, \psi)$, the Legendre transformation of the Lagrangian (the Hamiltonian) will be

$$\begin{aligned}\mathcal{L}(L) &= \boldsymbol{\nabla}_{\dot{q}} L \cdot \dot{\boldsymbol{q}} - L = \boldsymbol{\nabla}_{\dot{q}}(\tfrac{1}{2}\omega \cdot \boldsymbol{g}^{rb}) \cdot \dot{\boldsymbol{q}} + \boldsymbol{\nabla}_{\dot{q}}(\omega \cdot \boldsymbol{h}) \cdot \dot{\boldsymbol{q}} - \tfrac{1}{2}\omega \cdot \boldsymbol{g}^{rb} - \omega \cdot \boldsymbol{h} - T_r \\ &= \tfrac{1}{2}\omega \cdot \boldsymbol{g}^{rb} - T_r\end{aligned}$$

and since the relative kinetic energy is a function only of t, the Hamiltonian is

$$\mathcal{H} = \tfrac{1}{2}\omega \cdot \boldsymbol{g}^{rb}$$

that expressed in terms of the components of the angular momentum vector in the body frame coincides with equation (8), that is, $\mathcal{H} = T$,

$$\mathcal{H} = \tfrac{1}{2}(a_1 g_1^2 + a_2 g_2^2 + a_3 g_3^2) - (a_1 h_1 g_1 + a_2 h_2 g_2 + a_3 h_3 g_3). \tag{9}$$

The configuration space is $SO(3) \times T^3$. (NB Although the Hamiltonian has been derived in absence of external forces, the way followed still is valid when there are conservative forces; the Hamiltonian would be $\mathcal{H} = \frac{1}{2}\omega \cdot \boldsymbol{g}^{rb} + V$, with V the potential function.)

Instead of straightforwardly computing the conjugate moments, we will use the angular momentum to overlay the symplectic structure. The new moments Φ, Θ and Ψ will satisfy the differential identity

$$\boldsymbol{G} \cdot \mathrm{d}R = \Phi\, \mathrm{d}\phi + \Theta\, \mathrm{d}\vartheta + \Psi\, \mathrm{d}\psi.$$

Taking into account (1), there results that the moments are the projections of the total angular vector $\boldsymbol{G}$ onto the non orthonormal basis $\boldsymbol{s}_3, \boldsymbol{l}, \boldsymbol{b}_3$, resulting

$$\Phi = \boldsymbol{G} \cdot \boldsymbol{s}_3 \qquad \Theta = \boldsymbol{G} \cdot \boldsymbol{l} \qquad \Psi = \boldsymbol{G} \cdot \boldsymbol{b}_3.$$

Hence, by inversion, there results

$$\begin{aligned} g_1 &= \left(\frac{\Phi - \Psi \cos\vartheta}{\sin\theta} \right) \sin\psi + \Theta \cos\psi \\ g_2 &= \left(\frac{\Phi - \Psi \cos\vartheta}{\sin\theta} \right) \cos\psi - \Theta \sin\psi \\ g_3 &= \Psi. \end{aligned} \tag{10}$$

With these expressions, it is just a matter of computing derivatives to check that the Poisson brackets satisfy

$$\{g_1; g_2\} = -g_3 \qquad \{g_2; g_3\} = -g_1 \qquad \{g_3; g_1\} = -g_2. \tag{11}$$

This Lie–Poisson structure was already found in [20].

Thus, so far, we have that the Hamiltonian (9) of the problem is quadratic in a set of coordinates (g_1, g_2, g_3) that generate an $SU(2)$ algebraic structure. In these variables the system has a singular noncanonical bracket tensor, and a conserved quantity, called *Casimir*, exists due to the degeneracy of the bracket. The topology of the phase space is defined by a constant-energy sphere (Hopf sphere). This kind of problems, when the different parameters vary, have been studied in detail in [22–25].

Let us now introduce the nondimensional quantities

$$(\xi_1, \xi_2, \xi_3) = \frac{1}{G}(g_1, g_2, g_3) \qquad (\alpha_1, \alpha_2, \alpha_3) = \frac{1}{G}(h_1, h_2, h_3).$$

We may divide the Hamiltonian (9) by the non-zero quantity $G^2 = \|\boldsymbol{G}\|^2$. After this division, and with the new quantities, there results

$$\frac{1}{G^2}\mathcal{H} = \tfrac{1}{2}(a_1\xi_1^2 + a_2\xi_2^2 + a_3\xi_3^2) - (a_1\alpha_1\xi_1 + a_2\alpha_2\xi_2 + a_3\alpha_3\xi_3).$$

By making a transformation of the independent variable t to a new *time* τ by

$$\tau = \frac{1}{G^2}t$$

the Hamiltonian is converted into

$$\mathcal{K} = \tfrac{1}{2}(a_1\xi_1^2 + a_2\xi_2^2 + a_3\xi_3^2) - (a_1\alpha_1\xi_1 + a_2\alpha_2\xi_2 + a_3\alpha_3\xi_3) \tag{12}$$

and the variables (ξ_1, ξ_2, ξ_3) lie on the unit two-dimensional sphere $\mathcal{S}^2$

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = 1. \tag{13}$$

Most information about the phase flow comes from the equilibria. To find the equilibria, the first task is to obtain the equations of the motion. We did not express the Hamiltonian in terms of moments and coordinates, but instead, we have all the information to obtain the equations of the motion in terms of the components of the angular momentum. Indeed, by recalling that the time derivative of a function F along a Hamiltonian flow $\mathcal{H}$ is computed as $\mathrm{d}F/\mathrm{d}t = \{F; \mathcal{H}\}$, and taking into account the structure of the Poisson brackets (11) one easily has, by using the Hamiltonian (12),

$$\begin{aligned} \dot{\xi}_1 &= \{\xi_1; \mathcal{K}\} = (a_3 - a_2)\xi_2\xi_3 + a_2\alpha_2\xi_3 - a_3\alpha_3\xi_2 \\ \dot{\xi}_2 &= \{\xi_2; \mathcal{K}\} = (a_1 - a_3)\xi_1\xi_3 + a_3\alpha_3\xi_1 - a_1\alpha_1\xi_3 \\ \dot{\xi}_3 &= \{\xi_3; \mathcal{K}\} = (a_2 - a_1)\xi_1\xi_2 + a_1\alpha_1\xi_2 - a_2\alpha_2\xi_1. \end{aligned} \tag{14}$$

Our interest is focussed on studying the behaviour of the phase flow for different values of the internal moments (now α_i), that may be positive, negative or null. In this paper, we will consider the axial gyrostatic problem, that is, the case when only one of these parameters $\alpha_i \neq 0$. The remaining cases, either two or the three distinct of zero, will appear elsewhere.

4. Rotor with constant rotation around the smallest axis of inertia

In the case where the rotor has constant rotations about the smallest axis of inertia ($\boldsymbol{b}_1$), two of the components of the relative angular momentum, namely α_2 and α_3, vanish. By recalling that the variables ξ_i lie on the sphere $\mathcal{S}^2$, after a simple time scaling, the Hamiltonian is converted into

$$\mathcal{H} = \tfrac{1}{2}\xi_1^2 + \tfrac{1}{2}P\xi_2^2 + Q\xi_1 \tag{15}$$

where P and Q are dimensionless parameters defined by

$$P = \frac{(a_2 - a_3)}{(a_1 - a_3)} \qquad Q = \frac{-a_1\alpha_1}{(a_1 - a_3)}.$$

As it was remarked by Koiller [17], the metric (15) is left-invariant on the direct product $SO(3) \times S^1$.

The equations of the motion corresponding to this Hamiltonian are

$$\begin{aligned}\dot{\xi}_1 &= -P\xi_2\xi_3\\ \dot{\xi}_2 &= (Q + \xi_1)\xi_3\\ \dot{\xi}_3 &= -(Q + \xi_1 - P\xi_1)\xi_2.\end{aligned} \tag{16}$$

These equations present the symmetries

$$\begin{aligned}(\xi_2, t) &\longrightarrow (-\xi_2, -t)\\ (\xi_3, t) &\longrightarrow (-\xi_3, -t)\end{aligned}$$

which indicate that the phase flow is time reversal symmetric respect to the planes $\xi_2 = 0$ and $\xi_3 = 0$. Consequently, equilibria, if any, must lie in these planes.

The Hamiltonian (15) is invariant by the transformation $(\xi_1, Q) \longrightarrow (-\xi_1, -Q)$, which means that it is sufficient to consider only non-negative values of Q. Let us recall that for $Q = 0$ the rotor moment vanishes and the problem is reduced to the rigid body.

We might proceed to analyse the phase flow and the possible bifurcations directly from these equations of the motion. However, after all the transformations made, we manage to put the Hamiltonian exactly as it was studied by Lanchares and Elipe [25], with the difference that in our case, the parameter P is in the interval $0 < P < 1$, and that the symplectic structure associated to the variables on the sphere (u, v, w) defined in [25] was

$$\{u; v\} = w \qquad \{v; w\} = u \qquad \{w; u\} = v.$$

We may overcome this handicap by counting the time in the reverse sign. Anyway, for finding the equilibria, this fact has no influence since equilibria are obtained from the system (16) by making zero the right-hand side in it.

At the extrema of this interval, $P = 0$ and $P = 1$, the platform is axial symmetric ($a_2 = a_3$ or $a_2 = a_1$). These situations of axial symmetries will be analysed later on.

From the work of Lanchares and Elipe [25], for whatever P and $Q > 0$, the equilibria are those appearing on table 1 in a set of generic coordinates (u, v, w) on the unit sphere, that in the present case are related with ξ_i by the mapping

$$(u, v, w) \longrightarrow (\xi_1, \xi_2, \xi_3).$$

Table 1. Equilibria, existence and value of the energy at the equilibria when there is only one rotor.

Equilibrium point in coordinates (u, v, w)	Existence	Energy
$E_1 \equiv (1, 0, 0)$	Always	$\frac{1}{2} + Q$
$E_2 \equiv (-1, 0, 0)$	Always	$\frac{1}{2} - Q$
$E_3 \equiv \left(\frac{Q}{P-1}, \sqrt{1 - \frac{Q^2}{(P-1)^2}}, 0\right)$	$\lvert Q\rvert \leqslant \lvert P-1\rvert$	$\frac{-P + P^2 + Q^2}{2(-1+P)}$
$E_4 \equiv \left(\frac{Q}{P-1}, -\sqrt{1 - \frac{Q^2}{(P-1)^2}}, 0\right)$		
$E_5 \equiv \left(-Q, 0, \sqrt{1-Q^2}\right)$	$\lvert Q\rvert \leqslant 1$	$-\frac{1}{2}Q^2$
$E_6 \equiv \left(-Q, 0, -\sqrt{1-Q^2}\right)$		

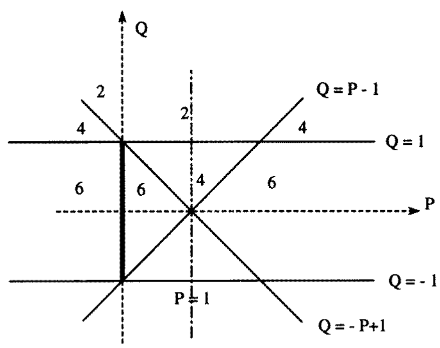

Figure 1. Partition in the parametric plane PQ for the Hamiltonian corresponding to only one rotor. For internal rotations about the largest axis of inertia $P < 0$; about the smallest one $0 < P < 1$, whereas for rotations about the intermediate axis of inertia, $P > 1$. The numbers 2, 4, 6 stand for the number of equilibria in each region of the partition.

The conditions of existence of these equilibria determine a partition (figure 1) in the parameter plane PQ, and at points belonging to the same region in the partition, the phase flow is similar, that is, the number of equilibria is the same, as well as their respective stability. The separatrices of the partition are the lines of parametric bifurcation.

Let P be a number in the open interval $(0, 1)$. Let us remind that $P = (a_2-a_3)/(a_1-a_3)$, which means that for a given gyrostat, this number is fixed; and let us study the evolution of the phase flow when the parameter $Q = -a_1\alpha_1/(a_1 - a_3)$ varies from $Q = 0$ (a rigid body) to a value $Q > 1$.

For $Q = 0$ the rigid body has six equilibria (the intersections of the positive and negative directions of the body frame with the sphere), four of them stable $E_{1,2}$, $E_{5,6}$ and two unstable $E_{3,4}$. From table 1, for $0 < Q < 1 - P$ there are still six points, four stable $E_{1,2} = (\pm 1, 0, 0)$ and $E_{5,6}$ that are on the meridian $\xi_1 = 0$, but migrating towards the point $(-1, 0, 0)$, and two unstable $E_{3,4}$, on the equator $\xi_3 = 0$ and moving towards $E_2 = (-1, 0, 0)$ as Q increases. Since the two unstable points have the same energy, these points are connected by heteroclinic orbits.

As soon as Q crosses the boundary $Q = 1 - P$, this configuration is broken, for the points $E_{3,4}$ merge into the point E_2, a pitchfork bifurcation occurs, and there are only four equilibria, three of them stable, $E_1 = (1, 0, 0)$ and $E_{5,6}$ whereas the point $E_2 = (-1, 0, 0)$ has changed its stability and now is unstable. Let us remember the *index theorem* for vector fields on manifolds: *the sum of the indices of the fixed points equals the Euler characteristic of the manifold* (see e.g. [26, 27]). In our case, the Euler characteristic of the sphere is two, and consequently, the number of stable points minus the number of unstable equilibria must be equal to two). From this unstable point two homoclinic orbits spring, each one encircling one of the stable points $E_{5,6}$.

As Q approaches the value $Q = 1$, these homoclinic orbits are collapsing towards the point E_2, and at $Q = 1$, the two equilibria $E_{5,6}$ coincide with E_2 that now becomes stable, through a pitchfork bifurcation. For $Q > 1$ the phase flow consists on rotations around the two stable equilibria E_1 and E_2.

5. Rotor with constant rotation around the largest axis of inertia

When the rotor is moving around the axis $\boldsymbol{b}_3$, the internal moments $\alpha_1 = \alpha_2 = 0$, and thus, the Hamiltonian (12) is reduced to

$$\mathcal{H} = \tfrac{1}{2}(a_1\xi_1^2 + a_2\xi_2^2 + a_3\xi_3^2) - a_3\alpha_3\xi_3.$$

Analogously to the preceding case, this Hamiltonian may be put in the form (15). Indeed, taking into account that the variables ξ_i lie on the unit sphere (13), we may eliminate ξ_2, and the Hamiltonian becomes

$$\mathcal{H} = \tfrac{1}{2}\left((a_1 - a_2)\xi_1^2 + (a_3 - a_2)\xi_3^2\right) - a_3\alpha_3\xi_3.$$

By defining the dimensionless parameters

$$P = \frac{a_3 - a_2}{a_1 - a_2} < 0 \qquad Q = \frac{-a_3\alpha_3}{a_1 - a_2}$$

and by making a time scaling transformation, the Hamiltonian for this case is

$$\mathcal{H} = \tfrac{1}{2}\xi_3^2 + \tfrac{1}{2}P\xi_1^2 + Q\xi_3 \tag{17}$$

that is to say it is equivalent to (15), since the variables appear in the same cyclic permutation, which is necessary because of the structure of their Poisson brackets.

The equations of the motion corresponding to this Hamiltonian are

$$\begin{aligned}
\dot{\xi}_1 &= (Q + \xi_3)\xi_2 \\
\dot{\xi}_2 &= -(Q + \xi_3 - P\xi_3)\xi_1 \\
\dot{\xi}_3 &= -P\xi_1\xi_2.
\end{aligned}$$

By defining now the mapping

$$(u, v, w) \longrightarrow (\xi_3, \xi_1, \xi_2)$$

the equilibria, the domain where they exist and the energy at them are those appearing on table 1.

To analyse the phase flow, we have to bear in mind that the parameter is now $P < 0$. From the partition of the parametric plane (figure 1), for a value of $P < 0$ and when the parameter Q (that is essentially the rotor moment) ranges from 0 to $Q > |1 - P|$, the phase portrait is completely analogous to the one described above for internal moment around the axis $\boldsymbol{b}_1$, but now, the bifurcations take place around the point $(0, 0, -1)$ rather than around the point $(-1, 0, 0)$.

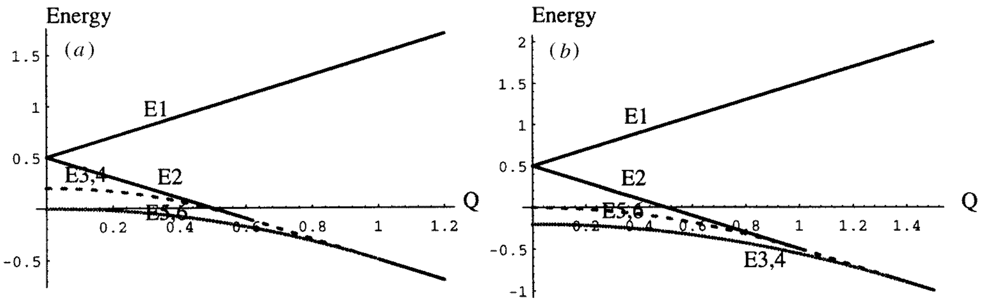

Figure 2. Energy plot at the equilibria for two cases (a) internal rotations about the smallest axis of inertia ($P = 0.4$) (b) internal rotations about the largest axis of inertia ($P = -0.4$). Broken curves represent unstable positions.

In the present case, for a given value of $P < 0$, when $0 < Q < 1$, the points $E_1 = (0, 0, 1)$, $E_2 = (0, 0, -1)$, $E_{3,4} = (\pm\sqrt{1 - Q^2/(P-1)^2}, 0, Q/(P-1))$ are stable, whereas the points $E_{5,6} = (0, \pm\sqrt{1 - Q^2}, -Q)$ are unstable, and since the energy at these unstable points is the same, they are connected by heteroclinic orbits. As Q approaches the value $Q = 1$, the unstable points are moving towards the point E_2, and at $Q = 1$, the three points $(E_{5,6}, E_2)$ merge into E_2 that becomes unstable, and a second pitchfork bifurcation occurs. For Q approaching the value $|1 - P|$, the stable points $E_{3,4}$ are close to the unstable equilibrium E_2, and for $Q = |1 - P|$, the three collapse into E_2, that becomes stable through a pitchfork bifurcation.

6. Rotor with constant rotation around the intermediate axis of inertia

When there are only rotor moments around the intermediate axis of inertia $\boldsymbol{b}_2$, the two remaining rotor moments are $\alpha_1 = \alpha_3 = 0$, and the Hamiltonian (12) is now reduced to

$$\mathcal{H} = \tfrac{1}{2}(a_1\xi_1^2 + a_2\xi_2^2 + a_3\xi_3^2) - a_2\alpha_2\xi_2.$$

Again, by using the constraint (13), we may eliminate ξ_1, and the Hamiltonian (after the corresponding time scaling) becomes

$$\mathcal{H} = \tfrac{1}{2}\xi_2^2 + \tfrac{1}{2}P\xi_3^2 + Q\xi_2 \tag{18}$$

where the dimensionless parameters P and Q are now

$$P = \frac{a_3 - a_1}{a_2 - a_1} > 1 \qquad Q = \frac{-a_2\alpha_2}{a_2 - a_1}.$$

This Hamiltonian (18) has the same form that the one studied by Lanchares and Elipe [25], simply by making

$$(u, v, w) \longrightarrow (\xi_2, \xi_3, \xi_1)$$

and restricting our analysis to the interval $1 < P < \infty$. The equilibria, and the domain where they are defined are those of table 1. The phase flow when the parameter Q evolves is depicted in figure 5.

From the partition of the parametric plane (figure 1), for a value of $P > 1$, there are either six, four or two equilibria in the phase flow, depending on which region the parameter Q is. When $Q < P - 1$ and $Q < 1$, there are six equilibria; four of them are stable: $E_{3,4} = (0, Q/(P-1), \pm\sqrt{1 - Q^2/(P-1)^2})$, $E_{5,6} = (\pm\sqrt{1 - Q^2}, -Q, 0)$; and

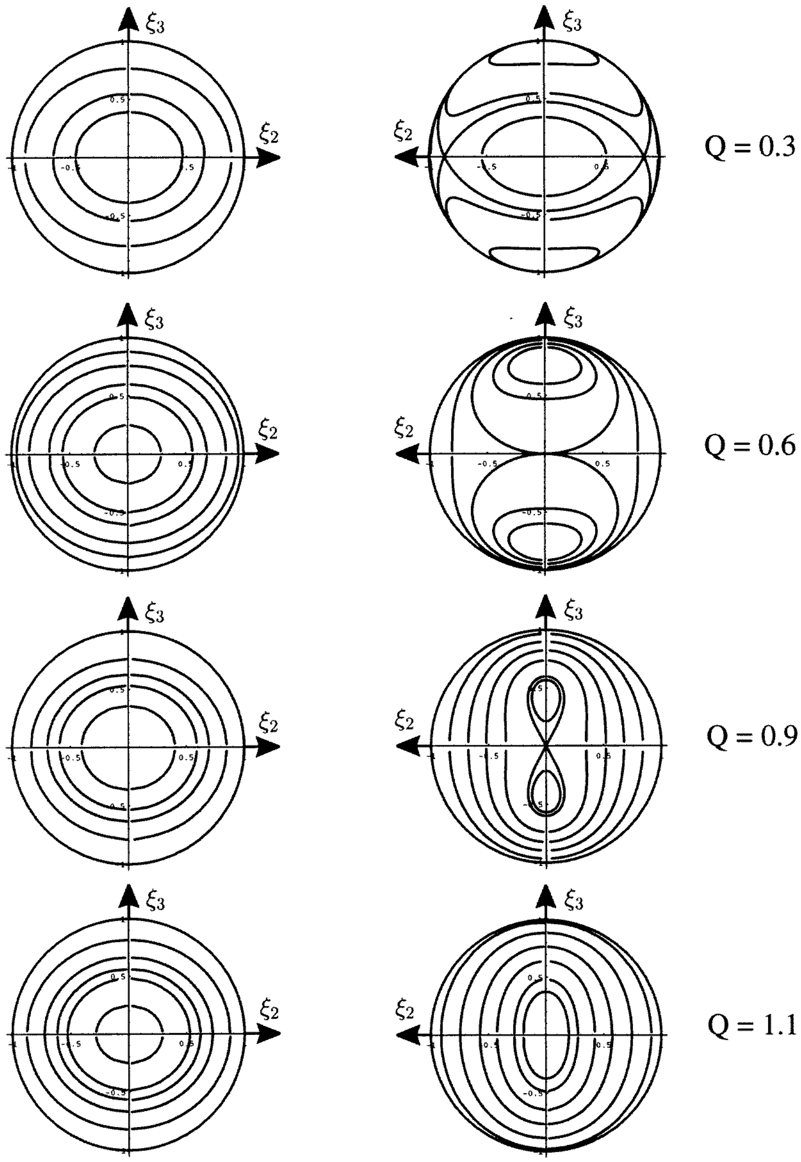

Figure 3. Phase portrait for internal rotations about the smallest axis of inertia ($a_1 = 1$, $a_2 = 0.7$, $a_3 = 0.5$ or $P = 0.4$) for several values of Q. The left column is the view from the point $E_1 = (1, 0, 0)$, the right column is the view from the point $E_2 = (-1, 0, 0)$. The phase flow evolution is made first through a pitchfork bifurcation at E_2 for $Q = 0.6$ and secondly through another pitchfork bifurcation at E_2 for $Q = 1$.

the two remaining $E_1 = (0, 0, 1)$ and $E_2 = (0, 0, -1)$ are unstable (let us recall the index theorem again). Since the energies at these two last points is different to one another, they are not connected by a heteroclinic orbit, but from the point E_1 emanate two homoclinic orbits encircling the points $E_{3,4}$ whereas from E_2 emanate two homoclinic orbits encircling the points $E_{5,6}$.

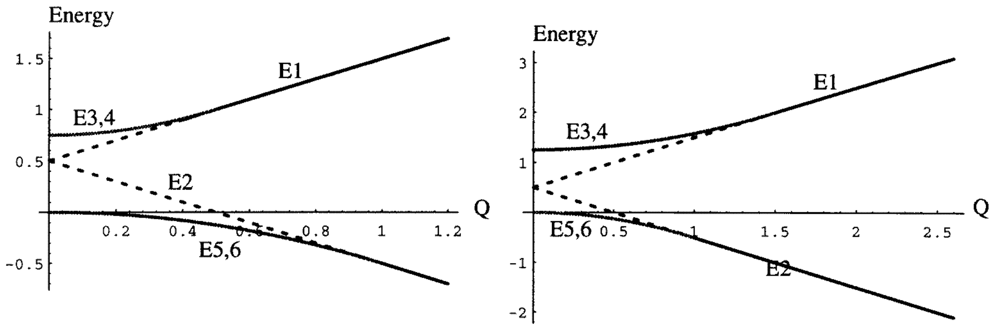

Figure 4. Energy plot at the equilibria for internal rotations about the intermediate axis of inertia (a) $P = 1.5$ and (b) $P = 2.5$. Broken curves represent unstable positions.

Let us remember that for $Q = 0$ (that is to say, for the rigid body problem) there are six points, four of them stable, the intersection of the sphere with the positive and negative directions of the axes $\boldsymbol{b}_1$ and $\boldsymbol{b}_3$ ($E_{5,6}$ and $E_{3,4}$ in our notation) and two unstable ($E_{1,2}$ at the extrema of $\pm\boldsymbol{b}_1$), joined by two heteroclinic orbits. As soon as $Q \neq 0$ this configuration is broken and the one described above appears, that we dubbed *tennis ball* configuration [28].

As Q increases, the lobes encircling the stable points shrink towards its respective unstable points: points $E_{3,4}$ move towards E_1 and points $E_{5,6}$ move towards E_2. Depending on the interval in which P is located one pair of the stable points will arrive before the other pair to the respective unstable point. If $1 < P < 2$, the points $E_{3,4}$ will gain this particular race and will collapse with E_1 that now becomes stable through a pitchfork bifurcation; when Q reaches the line $Q = 1$, the points $E_{5,6}$ disappear in E_2 through a second pitchfork bifurcation and now there are only two stable points, and the phase portrait is made on rotations around the points E_1 and E_2.

In contrast, if $P > 2$, the situation is the reverse, that is, the first pitchfork bifurcation occurs at the point E_2, for the $E_{5,6}$ have arrived first at $Q = 1$; the second pitchfork bifurcation occurs for $Q = P - 1 > 1$, where the stable points $E_{3,4}$ merge into E_1, that now becomes stable.

The case $P = 2$ is special, in the sense that the two pitchfork bifurcations take place simultaneously (for $Q = 1$), which could be deduced from the equations of the motion.

7. Cases of axial symmetry

It is well known that for the Euler Pointot motion of a rigid body axially symmetric, (for instance $a_1 = a_2 > a_3$), there are two isolated equilibria, namely the north and south poles, and besides, a dense set of equilibria, for all the points of the equator $\xi_3 = 0$ are equilibria; there is a degeneracy.

One can easily visualize this fact by recalling that the phase flow of this problem is made of the level contour of the energy ellipsoid (that is an ellipsoid of revolution)

$$\mathcal{H} = \tfrac{1}{2}a_1(\xi_1^2 + \xi_2^2) + \tfrac{1}{2}a_3\xi_3^2$$

on the momentum sphere

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = G^2.$$

The manifold

$$\tfrac{1}{2}a_1(\xi_1^2 + \xi_2^2) + \tfrac{1}{2}a_3\xi_3^2 = h = \tfrac{1}{2}a_1G^2$$

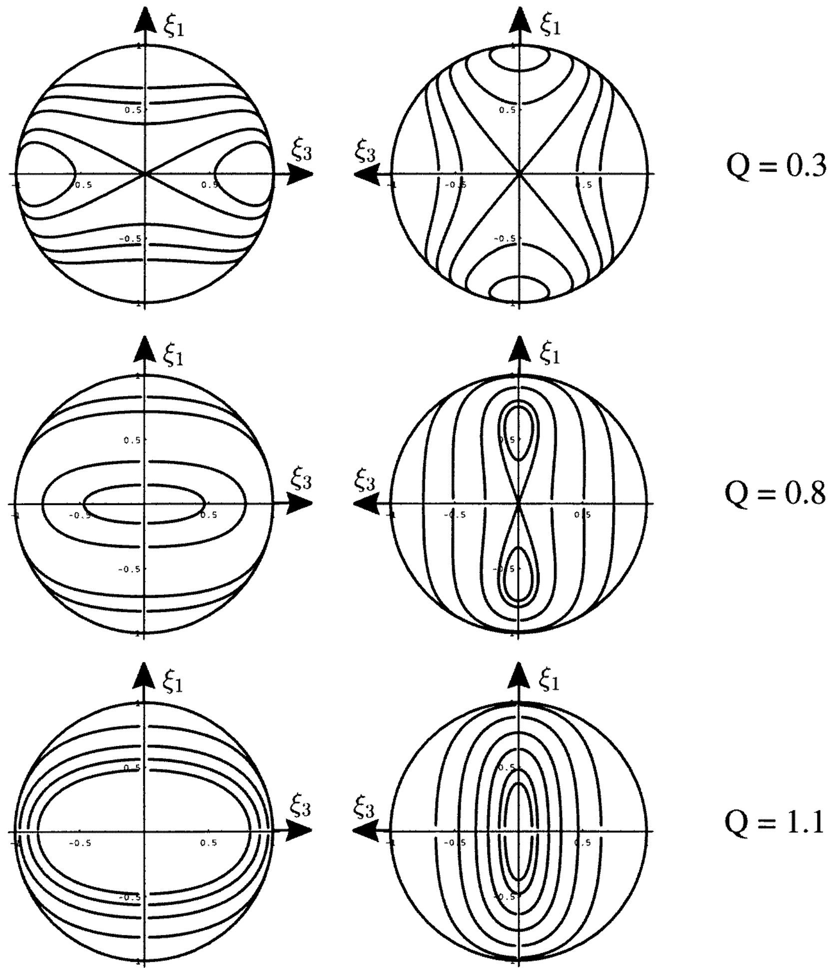

Figure 5. Phase portrait for internal rotations about the intermediate axis of inertia ($a_1 = 1$, $a_2 = 0.7$, $a_3 = 0.5$ or $P = 1.6666$) for several values of Q. The left column is the view from the point $E_1 = (0, 1, 0)$, the right column is the view from the point $E_2 = (0, -1, 0)$. The phase flow evolution is made through two pitchfork bifurcations, one at E_1 and the second one at E_2.

is tangent to the sphere along the equator $\xi_3 = 0$, thus, all points on this circle are equilibria.

The question now is whether the degeneracy will be preserved when a rotor spins about one of the principal axis of inertia. The answer is that indeed, this degeneracy is kept, but only when the rotor spins about the direction of the axis of symmetry of the gyrostat. Otherwise, the degeneracy is broken through an *oyster* bifurcation [29].

Again, the phase flow of an axially symmetric gyrostat in free rotation motion is made as the level contour of the energy ellipsoid (12) (where either $a_1 = a_2 > a_3$ or $a_1 > a_2 = a_3$ and only one of the $\alpha_i \neq 0$) and the momentum sphere (13).

However, by adding one spinning rotor, the centre of the ellipsoid of the revolution is shifted from the origin, that still is the centre of the sphere. Let us assume, for instance, that the axis of symmetry is the axis $\boldsymbol{b}_3$. In this case, the ellipsoid is

$$\mathcal{H} = \tfrac{1}{2}a_1(\xi_1^2 + \xi_2^2) + \tfrac{1}{2}a_3\xi_3^2 - a_3\alpha_3\xi_3.$$

This ellipsoid and the sphere are tangents along the small circle $\xi_3 = -a_3\alpha_3/(a-1-a_3)$, and therefore, each point belonging to this small circle is an equilibrium. Hence, the effect of the rotor consists of a displacement of the circle of degeneracy along the axis of symmetry.

When the rotor is spinning about another axis (either $\boldsymbol{b}_1$ or $\boldsymbol{b}_2$), the centre of ellipsoid of revolution is no longer the origin, but one point on the axis of symmetry, and consequently, the ellipsoid cannot be tangent to the sphere; therefore, the degeneracy is broken.

Let us see it in detail.

Appendix A. Case $a_1 = a_2 > a_3$

A.1. Rotor with constant rotation around the smallest axis of inertia

When $a_1 = a_2$, the Hamiltonian (15) is reduced to

$$\mathcal{H} = \tfrac{1}{2}(\xi_1^2 + \xi_2^2) + Q\xi_1$$

since now $P = (a_2 - a_3)/(a_1 - a_3) = 1$. The equations of the motion corresponding to this Hamiltonian are

$$\dot{\xi}_1 = -\xi_2\xi_3 \qquad \dot{\xi}_2 = (Q + \xi_1)\xi_3 \qquad \dot{\xi}_3 = -Q\xi_2.$$

For $Q = 0$, the points $E_{5,6} = (0, 0, \pm 1)$ are stable equilibrium points and besides, the equator $\xi_3 = 0$ is made of equilibria. However, as soon as $Q > 0$, the degeneracy is broken. Indeed, of the infinity equilibria on the equator, only two equilibria remain, namely the points $E_1 = (1, 0, 0)$ that is now stable, and $E_2 = (-1, 0, 0)$ that is unstable. The north and south poles start their migration towards the stable point E_2 along the meridian $\xi_2 = 0$, and now they are the stable points $E_{5,6} = (-Q, 0, \pm\sqrt{1 - Q^2})$. Thus, the degeneracy is broken through an *oyster* bifurcation; we may consider the two homoclinics as valves hinging on E_2, with the oyster closing its valves as Q tends to zero (see [29] for details).

As Q increases towards 1, the homoclinics and the equilibria inside them are approaching the unstable point $E_2 = (-1, 0, 0)$, and at $Q = 1$, a pitchfork bifurcation happens, the three points merge into E_2 that now is stable, and for $Q > 1$, the phase portrait is made of rotations around the axis $\boldsymbol{b}_1$.

A.2. Rotor with constant rotation around the intermediate axis of inertia

When $a_1 = a_2$, and $\alpha_1 = \alpha_3 = 0$, $\alpha_2 \neq 0$, the situation is analogous to the above studied. The Hamiltonian (18) is of no utility, for in this case $P = +\infty$ (one of the extrema of the definition domain of P), however, one can have the equations of motion straightforward from the equations (14):

$$\dot{\xi}_1 = -(\xi_2 + Q)\xi_3 \qquad \dot{\xi}_2 = \xi_1\xi_3 \qquad \dot{\xi}_3 = Q\xi_1$$

where now $Q = -a_2\alpha_2/(a_1 - a_3)$.

For $Q = 0$ the equilibria set is made of the points $E_{5,6} = (0, 0, \pm 1)$, that are stable, and of the equator $\xi_3 = 0$. Again, as soon as $Q > 0$, the degeneracy is broken through an oyster bifurcation, but now, the hinging point is $E_4 = (0, -1, 0)$ that is unstable; its opposite $E_3 = (0, 1, 0)$ is stable, and the stable points $E_{5,6} = (0, -Q, \pm\sqrt{1 - Q^2})$ tends towards the unstable point as Q increases. At $Q = 1$, the pitchfork bifurcation disappears and there are only two equilibria $E_{3,4}$ that are stable, and the phase flow is made of pure rotations around the axis $\boldsymbol{b}_2$.

A.3. Rotor with constant rotation around the largest axis of inertia

In this case, we cannot use the Hamiltonian (17), for now the parameter $P = -\infty$. However, from (14) and by imposing that $a_1 = a_2$, and $\alpha_1 = \alpha_2 = 0$, $\alpha_3 \neq 0$, we obtain the equations of motion directly from the equations (14):

$$\dot{\xi}_1 = -(\xi_3 + Q)\xi_2 \qquad \dot{\xi}_2 = (\xi_3 + Q)\xi_1 \qquad \dot{\xi}_3 = 0$$

where now $Q = a_3\alpha_3/(a_1 - a_3)$.

These equations have two stable equilibria $E_{5,6} = (0, 0, \pm 1)$, plus all the points of the minor circle $\xi_3 = -Q$. Thus, in the present case, the degeneracy of the rigid body (the equator $\xi_3 = 0$) is not broken as it was in the precedent cases, but it is shifted along the axis $\boldsymbol{b}_3$.

Appendix B. Case $a_1 > a_2 = a_3$

When the symmetry is such that $a_1 > a_2 = a_3$, the influence of the rotor is much analogous to the previous case considered, and we do not repeat here. Let us say that in this case, since the axial symmetry is along the axis $\boldsymbol{b}_1$, it is this axis who plays the role that the axis $\boldsymbol{b}_3$ had above. For rotor moment around the axis $\boldsymbol{b}_1$ the degeneracy is maintained, but shifted to the minor circle $\xi_1 = Q = a_1\alpha_1/(a_1 - a_3)$, whereas for internal moment around either $\boldsymbol{b}_2$ or $\boldsymbol{b}_3$, the degeneracy is broken through an oyster bifurcation.

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